Brownian particle having a fluctuating mass

M. Ausloos* and R. Lambiotte†
SUPRATECS, B5 Sart-Tilman, B-4000 Liège, Belgium
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We focus on the dynamics of a Brownian particle whose mass fluctuates. First we show that the behavior is similar to that of a Brownian particle moving in a fluctuating medium, as studied by Beck [Phys. Rev. Lett. **87**, 180601 (2001)]. By performing numerical simulations of the Langevin equation, we check the theoretical predictions derived in the adiabatic limit, i.e. when the mass fluctuation time scale is much larger than the time for reaching the local equilibrium, and study deviations outside this limit. We compare the mass velocity distribution with *truncated* Tsallis distributions [J. Stat. Phys. **52**, 479 (1988)] and find excellent agreement if the masses are chi-squared distributed. We also consider the diffusion of the Brownian particle by studying a Bernoulli random walk with fluctuating walk length in one dimension. We observe the time dependence of the position distribution kurtosis and find interesting behaviors. We point out a few physical cases, where the mass fluctuation problem could be encountered as a first approximation for agglomeration-fracture nonequilibrium processes.

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I. INTRODUCTION

It is well known that, while the Maxwell-Boltzmann distribution takes place in any system at equilibrium, nonequilibrium systems present in general qualitative and quantitative deviations from the former. A case of particular interest is that of distributions characterized by a power law tail, and therefore by an over-population for high energy as compared to the Gaussian. Indeed, such behaviors occur in a large number of physical systems, going from self-organized media to granular gases, and may have striking consequences due to the large probability of extreme/rare events.

The ubiquity of these *fat tail* distributions in nature has motivated several attempts in the literature to construct a general formalism for their description, one of the most recent being the Tsallis thermodynamic formalism [1,2]. The latter is based on a proper extremization of the nonextensive entropy,

$$S_q = k\frac{(1-\sum_i p_i^q)}{q-1}, \tag{1}$$

that leads to generalized canonical distributions, often called Tsallis distributions

$$f_q(z) = \frac{e_q^{-\beta' z}}{Z}. \tag{2}$$

In this expression, the variable z denotes the state of the system, and e_q^z is the q-exponential function defined by

$$e_q^z \equiv [1+(1-q)z]^{1/(1-q)}. \tag{3}$$

This definition implies that $e_1^z = e^z$, and that the energy is canonically distributed in the classical limit $q\to 1$, as expected. Let us also note that this formalism draws a direct parallelism with the equilibrium theory, where β' plays the role of the inverse of a temperature, and Z that of a partition function.

Tsallis distributions have been observed in numerous fields [3] but their fundamental origin is still debated due to the lack of a simple model and an exact treatment justifying the formalism. It is therefore important to study simple statistical models in order to show in which context Tsallis statistics apply. There are several microscopic ways to justify Tsallis statistics. One of them was introduced some years ago by Beck [1]. Another has been recently introduced by Thurner [4]. The Beck theory was first initiated by considering the case of a Brownian particle moving in a specially thermally fluctuating medium, i.e., the inverse temperature β is a chi-squared distributed random variable. Yet, Thurner [4] has shown that one can derive Tsallis distributions in a general way without the Beck "*chi-square* assumption."

In view of the above considerations, the motion of a Brownian particle whose *mass* is a random variable seems to be a paradigmatic example. Moreover, many physical cases are concerned by such situations. A sharp mass or volume variation of entities can be encountered in many nonequilibrium cases, ion-ion reaction [5–7], electrodeposition [8], granular flow [9–11] formation of planets through dust aggregation [12–14], film deposition [15], traffic jams [16,17], and even the stock markets [18,19] (in which the volume of exchanged shares fluctuates and the price undergoes some random walk). It should be noticed that *a priori* fluctuating mass problems differ from temperature fluctuations. Two masses can be addded to each other; this is hardly the case for temperatures. An exposé of such a generalized Brownian motion and the distinction between *masses* and *temperatures* will be emphasized in Secs. II and III.

By supposing two time scales, i.e., assuming that the relaxation processes for the particles are faster than the characteristic times for the mass fluctuation, it will be shown that

*Electronic address: Marcel.Ausloos@ulg.ac.be
†Electronic address: Renaud.Lambiotte@ulg.ac.be

its asymptotic velocity distribution is in general non-Maxwellian. Moreover, for some choice of the mass probability distribution, the Brownian particle velocities are Tsallis distributed. We verify this result by performing simulations of the corresponding Langevin equation.

Moreover, we also consider the case when the mass fluctuation is *not* adiabadically slow. We show in Sec. IV that the mass dependence of the relaxation rate may have non-negligible consequences on the velocity distribution of the Brownian particle. In order to fit the resulting distributions, and to describe the deviations from Tsallis statistics, we introduce truncated Tsallis distributions. Finally, in Sec. V, we study the diffusive properties of the Brownian particle. To do so, we model the motion of the particle by a random walk with time-dependent jump probabilities, associated with the mass fluctuations of the particle. The model exhibits standard diffusion, i.e., $\langle x^2\rangle \sim t$, but the shape of the scaling position distribution is anomalous and may exhibit quasistationary features. Section VI serves as a summary and conclusion.

II. GENERALIZING THE BROWNIAN MOTION TO ONE IN A FLUCTUATING MEDIUM

For setting the framework of the present paper, let us recall one microscopic way to justify Tsallis statistics [1]. Beck theory, later called *Superstatistics*, was first initiated by considering the case of a Brownian particle moving in a fluctuating medium, i.e., an ensemble of macroscopic particles evolving according to Langevin dynamics

$$m\partial_t v = -\lambda v + \sigma L(t), \tag{4}$$

where λ is the friction coefficient, σ describes the strength of the noise, and $L(t)$ is a Gaussian white noise [20]. Contrary to the classical Brownian motion, however, one may consider that the features of the medium may fluctuate temporally and/or spatially, namely the quantity $\beta=(\lambda)/(\sigma^2)$, i.e., a quantity which plays the role of the inverse of temperature, changes temporally on a time scale τ, or on the spatial scale L; see also [21]. For example, in his original paper, Beck assumed that this quantity fluctuates adiabatically slowly, namely that the time scale τ is much larger than the relaxation time for reaching the local equilibrium. In that case, the stationary solution of the nonequilibrium system consists in Boltzmann factors $e^{-\beta z}$ that are averaged over the various fluctuating inverse temperatures β,

$$f_{Beck}(z) = \frac{1}{K}\int d\beta g(\beta) e^{-\beta z}, \tag{5}$$

where K is a normalizing constant, and $g(\beta)$ is the probability distribution of β. Let us stress that ordinary statistical mechanics are recovered in the limit $g(\beta)\to\delta(\beta-\beta_E)$. In contrast, different choices for the statistics of β may lead to a large variety of probability distributions for the Brownian particle velocity.

Several forms for $g(\beta)$ have been studied in the literature [22], but one functional family of $g(\beta)$ is particularly interesting. Indeed, the generalized Langevin model Eq. (4) generates Tsallis statistics for the velocities of the Brownian particle if β is a chi-squared random variable,

$$g(\beta) = \frac{1}{b\Gamma(c)}\left(\frac{\beta}{b}\right)^{c-1} e^{-\beta/b}, \tag{6}$$

where b and c are positive real parameters which account for the average and the variance of β. Let us stress that a chi-squared distribution derives from the summation of squared Gaussian random variables X_i, $\beta=\Sigma_{i=1}^{2c}X_i^2$, where the X_i are independent, and $\langle X_i\rangle=0$. By introducing Eq. (6) into Eq. (5), it is straightforward to show that the velocity distribution of the particle is Eq. (2), if one identifies $c=1/(q-1)$ and $bc=\beta'$.

For completeness, let us also mention a study of Eq. (4) when λ fluctuates [23].

III. FLUCTUATING MASS

Let us now consider the diffusive properties of a macroscopic cluster such as one arising in granular media [9,10] or in traffic [16]. Such media are composed of a large number of macroscopic particles. Due to their inelastic interactions, the systems are composed of very dense regions evolving along more dilute ones. In general, there is a continuous exchange of particles between the dense cluster and the dilute region, so that the total mass of the macroscopic entity is not conserved. As a first approximation, we have thus considered the simplest approximation for this dynamics, namely the cluster is one Brownian-like particle whose mass fluctuates in the course of time. To mimic this effect, we have assumed that (i) the distribution of masses is *a priori* given by $g(m)$, and (ii) the mass of the cluster fluctuates with a characteristic time τ. By definition, this model evolves according to the Langevin equation (4), where m is now the random variable.

Given some realization of the random mass, say $m=m_R$, one easily checks that the velocity distribution of the particle converges toward the distribution

$$f_B(v) \to \frac{\sqrt{\beta m_R}}{\sqrt{2\pi}} e^{-\beta m_R v^2/2}. \tag{7}$$

This relaxation process takes place over a time scale $t_R \sim m_R/\lambda$. Therefore, if the separation of time scales $t_R \ll \tau$ applies, the asymptotic velocity distribution of the cluster is given by

$$f_B(v) = \int dm g(m) \frac{\sqrt{\beta m}}{\sqrt{2\pi}} e^{-\beta m v^2/2}. \tag{8}$$

Consequently, this leads to a Tsallis distribution if the masses are chi-squared distributed. In that sense there is a direct correspondence between the Beck approach and ours. However, there is more to see in the latter case because it justifies the Tsallis nonextensive entropy approach in a more mechanistic way. Subsequently two basic questions can be raised; (i) what are the limits of validity of such a nonequilibrium approach? and (ii) are the mass fluctuation time scales observable?

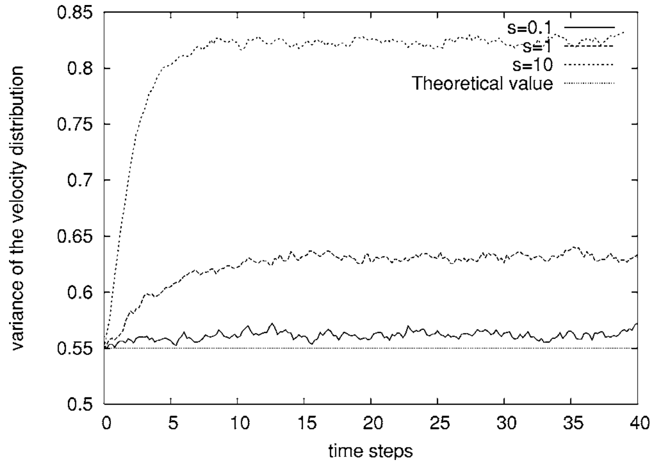

FIG. 1. Time evolution of the $\langle v^2\rangle$ for three values of τ, characterized by $s \equiv t_R/\tau = [0.1, 1, 10]$. The possible masses in the system are 1 and 10. Consequently, the $\langle v^2\rangle$ for each species are respectively 1.0 and 0.1. Moreover, in the slow limit ($\tau \gg t_R$), the asymptotic velocity fluctuation is, on average, 0.55. When τ is small, most of the particle velocities are distributed like the light ones, and their energy is closer to 1.0 than to 0.55.

IV. RELAXATION MECHANISMS

In this section, we report on numerical simulations of the random process Eq. (4). The objective is twofold. First, we verify the theoretical prediction Eq. (8) in the adiabatic limit. Next, this allows one to study systems which are beyond the range of the validity of Eq. (8), namely systems, where the separation of scales $t_R \ll \tau$ does not apply, thereby investigating the effects of competition between the relaxation to equilibrium and the fluctuating features of this equilibrium state. This program is achieved by considering three different relaxation characteristic time scales for the processes, namely $s = t_R/\tau = (0.1,\ 1.0,\ 10)$. Moreover, we first consider a paradigmatic case when the masses can switch between two different discrete values, each with equal probability,

$$g(m) = \frac{1}{2}[\delta(m - m_1) + \delta(m - m_2)]. \tag{9}$$

If Eq. (8) applies, i.e., in the slow fluctuation limit, it is straightforward to show [21] that Eq. (9) leads to

$$f_B(v) = \frac{1}{2}\sqrt{\frac{\beta}{2\pi}}(\sqrt{m_1}e^{-\beta m_1 v^2/2} + \sqrt{m_2}e^{-\beta m_2 v^2/2}). \tag{10}$$

Before focusing on the velocity distributions for the Brownian particles, let us stress that their average energy depends on the speed of the fluctuation mechanism. Indeed, in the slow limit $s \ll 1$, the energy of the cluster converges very rapidly toward the equipartition value $m_i\langle v^2\rangle = e$, where e is the average kinetic energy of the bath and m_i is the mass of the cluster at that time. This implies that the fluctuations of the measured velocities are given by $\langle v^2\rangle = (\frac{1}{2})(e/m_1 + e/m_2$. In contrast, in the faster limit $s > 1$, the dependence of the characteristic time $t_R \sim m/\lambda$ cannot be neglected. Indeed, this relation implies that particles with a smaller mass relax faster than particles with a larger mass. Consequently, the lighter particles should have a value close to their equipartition value $m_i\langle v^2\rangle = e$, while the heavier particles should have an energy larger than their expected value. This property is verified in Fig. 1.

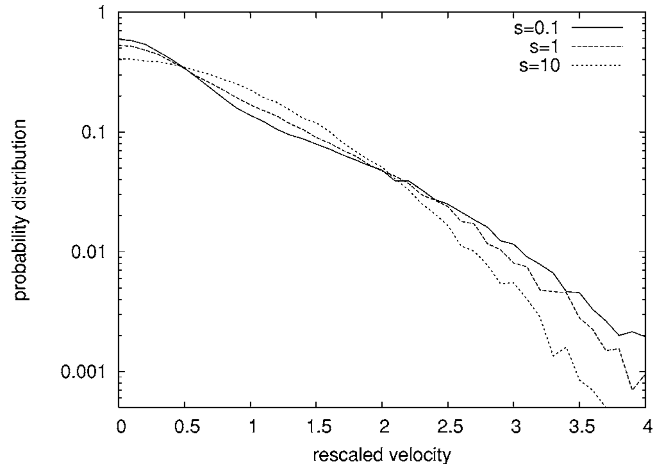

FIG. 2. Asymptotic velocity distribution for different relaxation time scales s of a Brownian particle, namely 0.1, 1, and 10. The possible masses in the system are 1 and 10. The velocities are rescaled so that $\langle v^2\rangle = 1$.

In order to compare the velocity distributions for different time scales τ, and therefore at different energies, we rescale the velocities so that $\langle v^2\rangle = 1$. The results, as plotted in Fig. 2, confirm the theoretical predictions Eq. (10) and our description in the previous paragraph. Indeed, in the slow limit $s = 0.1$, the velocity distribution converges toward the leptokurtic distribution Eq. (10), i.e., a distribution with a positive kurtosis and an overpopulated tail. In contrast, when the mass dependence of the t_R has to be taken into account, most of the particles have velocities distributed like those of the light particles, and namely are Maxwell-Boltzmann distributed.

In the case of the general Tsallis distributions (see Fig. 3) obtained through chi-squared distributed cluster masses, the problem is more complex due to the continuum of masses in the system, and to the associated continuum of characteristic relaxation times. Moreover, the existence of extreme values for the masses may cause nonrealistic numerical problems. Indeed, arbitrary small masses lead to arbitrary high values of the velocities. In the following, we avoid this effect that is responsible for the power law tails observed in the Tsallis distributions. This is justified by the fact that any physical system has a minimum size for its internal components. Similarly, we restrict the maximum size of the clusters in order to avoid infinitely slow relaxation processes. These limitations are formalized by using the following truncation for the mass distributions, inspired by truncated Lévy distributions [24]:

$$g(m;c) = k\chi^2(m;c) \text{ if } m > a \text{ and } m < b,$$

$$g(m;c) = 0 \text{ otherwise}, \tag{11}$$

where c is a parameter characterizing the chi-squared distribution, k is a normalizing constant, and $a < b$ are cutoff pa-

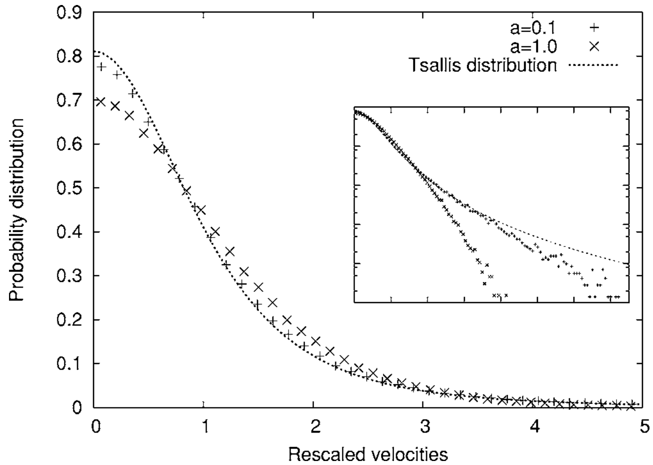

FIG. 3. Truncated Tsallis distribution $g_2(m)$ for $a=0.1$, $a=1.0$, and $b=\infty$ in rescaled velocities, so that $\langle|v|\rangle=1$. The solid line is the Tsallis distribution $(8/\pi^2)(1)/([1+(2/\pi)^2v^2]^2)$. The cut-off procedure gives more weight to the central range of the distribution ($m\in[1,3]$), and decreases the importance of the peak and of the tail. The inset corresponds to these functions in logarithamic-normal scale, in the interval [0,15].

rameters. In the following, we use $a=0.01$ and $b=100.0$ in the simulations of the Langevin equation. As shown in Fig. 4, this procedure is a natural way to smooth the tail of the Tsallis velocity distribution while preserving its core. This method, that will be discussed further in a forthcoming paper, should be applicable to a large variety of problems (like those mentioned in the Introduction) where extreme events have to be truncated for physical reasons.

Numerical simulations of the Langevin equation for chi-squared mass distributions generalize in a straightforward way the results obtained from the two-level distribution in Eq. (10). Indeed, the faster the mass distribution fluctuates the larger the deviations from the Tsallis distribution are. Moreover, these deviations have a tendency to underpopulate the tail of the distribution, and therefore to avoid the realization of extreme values of the random process.

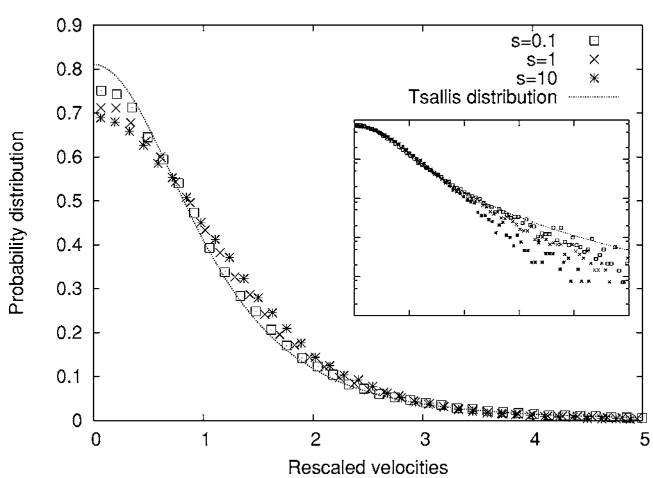

FIG. 4. Asymptotic solutions of the Langevin equation for three time scales, where we fix the velocity scale $\langle|v|\rangle=1$. The solid line is the theoretical distribution obtained from (8).

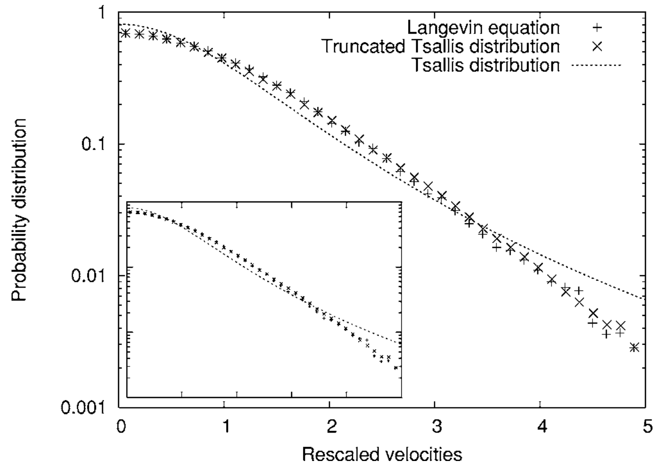

FIG. 5. Stationary solution of the Langevin equation for $s=10$ and truncated Tsallis distribution for $a=1$. The dotted line is the Tsallis distribution $(8/\pi^2)(1/([1+(2/\pi)^2v^2]^2)$.

It is also important to note the similarities between these asymptotic solutions of the Langevin equation and the truncated Tsallis distributions defined by Eq. (11). They highlight the flexibility of truncated Tsallis distributions in order to describe deviations from the Tsallis distributions, as shown in Fig. 5.

V. DIFFUSION

In this last section, we focus on the diffusive propertiesj of a Brownian particle with fluctuating mass [25,26]. Therefore, we take into account the hydrodynamic time scale t_H, that is associated to the evolution of spatial inhomogeneities. It is well known [27], when the mass of the Brownian particle is constant in time, that a separation of time scales $t_H \gg t_R$ is required in order to derive the diffusion equation

$$\partial_t n(\mathbf{x};t) = D\partial_x^2 n(\mathbf{x};t), \tag{12}$$

where D is fixed by the Einstein formula.

When the mass of the Brownian particle fluctuates, however, we have discussed above that there is an additional time scale τ in the dynamics. As a first approximation, we restrict the scope to the limit of very slow fluctuations ($\tau \gg t_H$), that is more restrictive than the limit discussed in the previous section ($\tau \gg t_R$). In that case, it is possible to show [28] that the Chapman-Enskog procedure leads to Eq. (12), where D is now time-dependent, i.e., a random variable that is a function of the mass of the Brownian particle.

In the following, we investigate the process associated to Eq. (12) with the fluctuating diffusion coefficient. To do so, we simplify the analysis by considering a one-dimensional discrete time random walk, where the jump probabilities may fluctuate in time [29]. Namely, the walker located at x performs at each time step a jump of length l, with Bernouilli probabilities,

$$P\,(k)\big|_l = \tfrac{1}{2}[\delta(k,l) + \delta(k,-l)]. \tag{13}$$

The quantity l fluctuates between two integer values $l_A \leq l_B$ that correspond to a heavy/light state for the Brownian par-

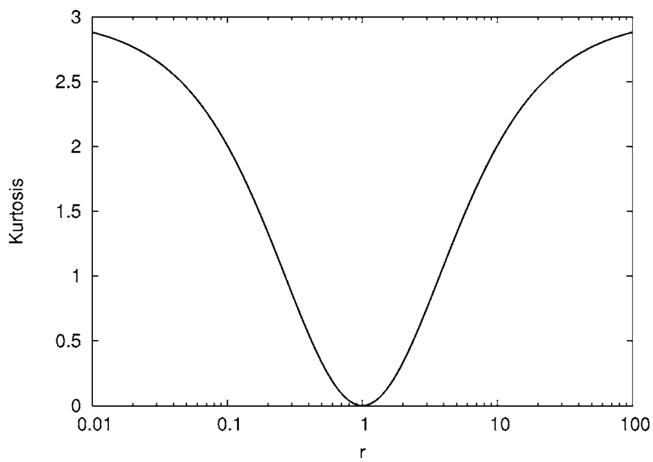

FIG. 6. Kurtosis of the distance distribution Eq. (16), for two Bernouilli random walkers in one-dimension with different walk lengths.

ticle. When $l_A = l_B$, it is easy to show that the first distance moments $m_i = \langle x^i \rangle$ asymptotically behave as

$$m_2 = Dt,$$

$$m_4 = 3D^2 t^2, \tag{14}$$

where the diffusion coefficient $D = l_A^2$. Relation (14) implies that the kurtosis of the distance distribution $\kappa = (m_4)/(m_2^2) - 3$ vanishes asymptotically, as required by the Central-Limit Theorem.

Another simple limit consists in a system with $l_A \neq l_B$, and where the mass fluctuation process is infinitely slow. Consequently, the particles do not change mass and the system is composed of two species that diffuse differently. In that case, the first distance moments read,

$$m_2 = \tfrac{1}{2}(D_A + D_B)t,$$

$$m_4 = \tfrac{3}{2}(D_A^2 + D_B^2)t^2, \tag{15}$$

where the diffusion coefficients are $D_A = l_A^2$ and $D_B = l_B^2$. This relation implies that the asymptotic kurtosis is equal to

$$\kappa = 6\frac{(D_A^2 + D_B^2)}{(D_A + D_B)^2} - 3. \tag{16}$$

In Fig. 6, we plot $\kappa(r) = 6[(1+r^2)/(1+r)^2] - 3$, where $r = D_A/D_B$. The figure shows that $\kappa \geqslant 0$, i.e., the distribution is characterized by an overpopulated tail, except in the usual limit $r = 1$. This result is expected, as a system with $r \neq 1$ is composed of two species that explore the space at different speeds. Let us stress that despite this anomalous position distribution, the diffusion is standard, i.e., $\langle x^2 \rangle \sim t$ [see Eq. (15)].

In order to study intermediate situations, we have performed computer simulations of the random walk. The system is composed of 50 000 walkers, that are initially located at $x = 0$ and randomly divided in the species A/B. The mass fluctuations are uncorrelated and occur with the probability

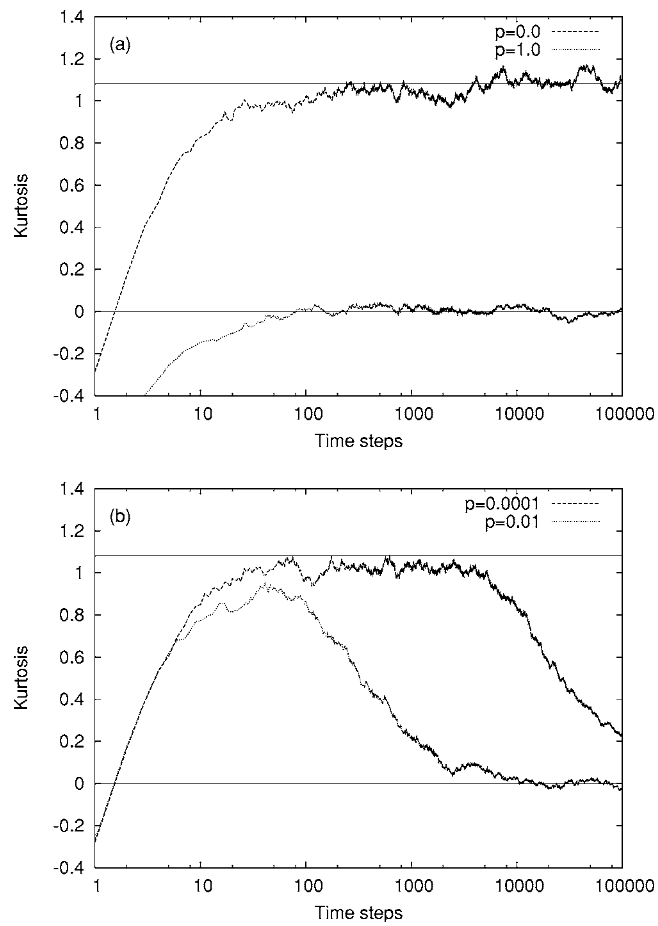

FIG. 7. Kurtosis κ of the distance distribution, as a function of time, for the Bernouilli random walker with fluctuating mass (see the text). In (a), we focus on the limiting cases $p = 0.0$ and $p = 1.0$ that converge toward the theoretical prediction, Eq. (16), $\kappa = 1.08$, and $\kappa_G = 0$ (Gaussian), respectively. These asymptotic values are represented by solid lines. In (b), we focus on the intermediate cases $p = 0.01$ and $p = 0.0001$. The state associated with Eq. (16) is stationary during a long time extent, and is followed by a convergence toward the Gaussian.

$p \in [0,1]$ at each time step, i.e., the characteristic time of the fluctuations is $\tau \sim p^{-1}$. In the simulations, we have used $l_A = 1$ and $l_B = 2$, so that the prediction for the kurtosis in the limit $\tau \to \infty$ is $\kappa_{AB} = 1.08$. This prediction [Fig. 7(a)] is verified by simulations in a system, where the mass of a walker is constant in time. In the limit $\tau = 1$, i.e., the walker changes mass at each time step, the kurtosis converges rapidly toward the Gaussian value $\kappa_G = 0$. In contrast, for higher finite values of τ [Fig. 7(b)], one observes a crossover between the two asymptotic behaviors κ_{AB} and κ_G. The stability of the state κ_{AB} is longer and longer for increasing values of τ. This quasistationarity [30] originates from the following reason. Over a long time period T, with $T \gg \tau$, the particles have suffered so many changes from state A to B and soforth, that their asymptotic dynamics is equivalent to that of a random walk with two possible jump lengths l_A and l_B, with the probabilities,

$$P(k) = \frac{1}{4}[\delta(k,l_A) + \delta(k,-l_A) + \delta(k,l_B) + \delta(k,-l_B)]. \quad (17)$$

Consequently, its asymptotic dynamics is subject to the classical Central-Limit Theorem, and the position distribution is a Gaussian for $T \gg \tau$.

VI. CONCLUSIONS

In examining an appartently unusual generalization of the old Brownian motion problem, i.e., a particle with fluctuating mass, we have found a very simple example justifying nonextensive thermodynamics. However, the simplicity is related to underlying considerations on quite various physical (or other) systems in which some "mass" is evolving with time, sometimes stochastically, as mentioned in the introduction of this paper.

In the first approach above, the mass statistics and their time evolution have been chosen simply *a priori*. A more detailed study should require (i) a generalization of the mass distributions, (ii) of their time evolution, and (iii) a dynamical treatment of the fluctuating mass by Langevin equations. This additional modeling depends on the nature of the considered problem.

Given these approximations, the velocity distribution of the Brownian particle has been studied by performing simulations that highlight the important role of the mass fluctuation time scale. In the case of chi-squared mass distribution, it is shown that *truncated* Tsallis distributions seem to describe in a relevant way deviations from the Tsallis statistics. Such distributions should be applicable to a large variety of problems, where extreme events have to be truncated for physical reasons, e.g., finite size effects,—when there is no infinity! Among these possible applications, let us note their occurrence in airline disasters statistics [31].

We have also studied the diffusive properties of the Brownian particle. In the limit of slow mass fluctuation times, the particle motion is modeled by a random walk with time-dependent jump probabilities. Moreover, for the sake of clarity, we restrict the scope to a dichotomous mass distribution. This modeling is a simplification of the complete problem, that has to be justified by a complete analysis starting with the Fokker-Planck equation itself [31]. Nonetheless, despite its apparent simplicity, the random walk analysis shows nontrivial behaviors, namely the system is characterized by standard diffusion, associated with non-Gaussian scaling distributions and quasistationary solutions. These features originate from very general mechanisms that suggest their relevance in various systems with fluctuating "mass" parameters.

In conclusion, we have generalized the Brownian motion to the case of fluctuating mass systems, and distinguish them from systems in which there are local temperature fluctuations. We have found that the velocity of such a particle may exhibit various anomalous distributions, including Tsallis distributions and truncated Tsallis distributions. The study of a one-dimensional random walk also indicates anomalies in the kurtosis of the position distributions which might indicate physical biases in various processes as those recalled in the Introduction. This suggests one look for the value of higher order distribution moments and time evolution for an understanding of the properties of the nonequillibrium systems.

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